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On the classical limit and the problem of phase transitions

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Abstract. Some aspects of the relation between proofs of both absence and existence of phase transitions for a class of quantum spin systems and their classical counterparts are discussed. The results of absence of phase transitions apply to a large class of classical systems with special symmetry already considered by Vuillermot and Romerio, and to their quantum mechanical analogues.

1. Introduction and summary

Recent great progress in the subject of phase transitions (Fröhlich *et al* 1976a, b, Dyson *et al* 1976, 1977) also showed a complete analogy between proofs of existence of a phase transition in the quantum mechanical isotropic Heisenberg model with nearest-neighbour interactions (Dyson *et al* 1976, 1977) and its classical counterpart (Fröhlich *et al* 1976a, b). In fact, the strategy adopted for the quantum case by Dyson *et al* (1976, 1977) was the same as that adopted by Fröhlich *et al* (1976a, b) for the classical model, the essential difference being the replacement in the basic inequality of Fröhlich *et al* (1976a, b) of the classical scalar product by the Bogoliubov (1962) scalar product.

In this paper, we discuss this correspondence in greater detail. In § 2 we study the classical limit of the Bogoliubov scalar product for a class of operators along the lines of Lieb (1973). In § 3 we discuss the relation between the classical (Mermin 1967, Vuillermot and Romerio 1975, Romerio and Vuillermot 1974) and quantum mechanical (Mermin and Wagner 1966) proofs of the absence of phase transitions for a large class of classical systems with special symmetry, already considered in Vuillermot and Romerio (1975), and to their quantum mechanical counterparts. Section 4 contains some brief remarks on the relation between existing proofs (Fröhlich *et al* 1976a, b, Dyson *et al* 1976, 1977) of phase transitions for the classical and quantum Heisenberg models.

2. On the classical limit of the Bogoliubov scalar product

An important role in the understanding of the relation between classical and quantum proofs of the absence or existence of phase transitions is played by the classical limit of the Bogoliubov scalar product, which we now discuss along the lines of Lieb (1973).

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Let H_Λ be the Hamiltonian of a quantum spin system of spin S for a finite region $\Lambda \subset \mathbb{Z}^3$ on $\mathcal{H}_\Lambda = \otimes_{i=1}^{|\Lambda|} \mathbb{C}_i^{2S+1}$, $|\Lambda|$ being the number of sites in Λ. H_Λ may be an arbitrary polynomial on the spin operators $\boldsymbol{S}_i$, $i = 1, \ldots, |\Lambda|$, but it is required (Lieb 1973) to be linear in the operators $\boldsymbol{S}_i$ of each spin. For A, B any two operators on $\mathcal{H}_\Lambda$ and any $0 < \beta < \infty$, we define the Bogoliubov scalar product of A and B as being the quantity

$$(A, B)_\Lambda^\beta \equiv \int_0^\beta d\lambda \, \langle e^{\lambda H_\Lambda} A^* e^{-\lambda H_\Lambda} B \rangle_\Lambda^\beta \tag{2.1}$$

where, for any matrix A on $\mathcal{H}_\Lambda$,

$$\langle A \rangle_\Lambda^\beta \equiv \mathrm{Tr}_{\mathcal{H}_\Lambda}(e^{-\beta H_\Lambda} A)/\mathrm{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_\Lambda}. \tag{2.2}$$

By the 'classical operator corresponding to a particular operator A on $\mathcal{H}_\Lambda$, linear in each of the spins $\boldsymbol{S}_i$', we mean the operator A_c on $\mathcal{H}_\Lambda^c \equiv \otimes_{i=1}^{|\Lambda|} L^2(\mathcal{S}_i, d\mu_i)$, $\mathcal{S}_i$ being a copy of the unit sphere in $\mathbf{R}^3$ and μ_i a copy of the usual measure on $\mathcal{S}_i$ (Lieb 1973), obtained from A by replacing each spin operator $\boldsymbol{S}_i$ in A by a classical unit vector $\boldsymbol{t}_i$ in $\mathcal{S}_i$. By A_S we denote the operator obtained from A by replacing each spin operator $\boldsymbol{S}_i$ in A by $\boldsymbol{S}_i/S$. We need the following lemma, which is a simple consequence of Lieb's (1973) inequalities.

Lemma 2.1. Let A be any operator on $\mathcal{H}_\Lambda$ linear in the operators $\boldsymbol{S}_i$ of each spin, and let $\langle \cdots \rangle_\Lambda^{\beta,S}$ denote the thermal expectation value as in (2.2), but replacing H_Λ by $H_{\Lambda,S}$. Then for each fixed $\Lambda \subset \mathbb{Z}^3$:

$$\lim_{S\to\infty} \langle A_S \rangle_\Lambda^{\beta,S} = \langle A_c \rangle_\Lambda^{\beta,c} \tag{2.3}$$

where $\langle A_c \rangle_\Lambda^{\beta,c}$ denotes the thermal expectation value of the classical operator A_c corresponding to A in the ensemble defined by the classical Hamiltonian $H_{\Lambda,c}$ corresponding to H_Λ.

Proof. We have Lieb's (1973) inequalities

$$\lambda^{-1}(f_\Lambda^c(0;1) - f_\Lambda^c(-\lambda, \delta_S)) \geq \langle A_S \rangle_\Lambda^{\beta,S} \geq \lambda^{-1}(f_\Lambda^c(\lambda;\delta_S) - f_\Lambda^c(0;1)) \qquad \forall \lambda \in \mathbf{R}_+ \tag{2.4a}$$

where $\delta_S = (S+1)/S$ and

$$f_\Lambda^c(\lambda;\delta) \equiv -\beta^{-1} \ln \int \bigotimes_{i=1}^{|\Lambda|} d\mu_i(\Omega_i) \exp[-\beta(H_{\Lambda,c}^\delta + \lambda A_c^\delta)] \tag{2.4b}$$

and where $\Omega_i \equiv (\theta_i, \varphi_i)$, $0 \leq \theta_i < \pi$, $0 \leq \varphi_i < 2\pi$, $d\mu_i(\Omega_i) = \sin\theta_i \, d\theta_i \, d\varphi_i$, $i = 1, \ldots, |\Lambda|$, and $H_{\Lambda,c}^\delta$ (and similarly A_c^δ) is obtained from $H_{\Lambda,c}$ by multiplying each classical spin unit vector in $H_{\Lambda,c}$ by δ. From (2.4*a*) taking the limit $S \to \infty$ we obtain

$$\lambda^{-1}(f_\Lambda^c(0;1) - f_\Lambda^c(-\lambda;1)) \geq \lim_{S\to\infty} \langle A_S \rangle_\Lambda^{\beta,S} \geq \lambda^{-1}(f_\Lambda^c(\lambda;1) - f_\Lambda^c(0;1)) \qquad \forall \lambda \in \mathbf{R}_+. \tag{2.5}$$

Now, for Λ fixed $f_\Lambda^c(\lambda; 1)$ is differentiable in λ for $\lambda \in \mathbf{R}$ and its derivative at $\lambda = 0$ is $\langle A_c \rangle_\Lambda^{\beta,c}$. Hence (2.5) implies (2.3).

Proposition 2.1.

$$\lim_{S\to\infty} (A_S, B_S)_\Lambda^{\beta,S} = \beta \langle A_c, B_c \rangle_\Lambda^{\beta,c} \tag{2.6}$$

where $(A, B)_\Lambda^{\beta,S}$ is obtained from (2.1) by the replacement of H_Λ by $H_{\Lambda,S}$, and

$$\langle A_c, B_c\rangle_\Lambda^{\beta,c} \equiv \int \prod_{i=1}^{|\Lambda|} \mathrm{d}\mu_i(\Omega_i)\bar{A}_c(\Omega)B_c(\Omega) \tag{2.7}$$

where Ω is the Cartesian product of the Ω_i, $i = 1, \ldots, |\Lambda|$, A_c and B_c denote functions of Ω and the bar denotes complex conjugate.

Remark. To relate this to the more general setting of § 3, we note that the group above is always the Cartesian product indexed by the points in Λ of copies of $G = \mathrm{SO}(3)$. If $K = \mathrm{SO}(2)$ is the isotropy subgroup of a point in the unit sphere, each measure μ in (2.4*b*) is a copy of the normalized measure induced by the Haar measure on the homogeneous space G/K. The scalar product in (2.7) is effectively a special case of the one considered in § 3.

Proof. By (2.1)

$$(A_S, B_S)_\Lambda^{\beta,S} = \int_0^\beta \mathrm{d}\lambda\ \langle e^{\lambda H_{\Lambda,S}}[A_S^*, \mathrm{e}^{-\lambda H_{\Lambda,S}}]B_S\rangle_\Lambda^{\beta,S} + \beta\langle A_S^*, B_S\rangle_\Lambda^{\beta,S}. \tag{2.8}$$

Although (A_S^*, B_S) may involve quadratic terms in the spin at a certain site of type (S_i^2), it may easily be proved that lemma 2.1 is applicable and yields

$$\lim_{S\to\infty} \langle A_S^*, B_S\rangle_\Lambda^{\beta,S} = \langle A_c^*, B_c\rangle_\Lambda^{\beta,c}. \tag{2.9}$$

Further

$$[A_S^*, \mathrm{e}^{-\lambda H_{\Lambda,S}}] = -\int_0^\lambda \mathrm{d}x\, \mathrm{e}^{-(\lambda-x)H_{\Lambda,S}}[A_S^*, H_{\Lambda,S}]\, \mathrm{e}^{-xH_{\Lambda,S}} \tag{2.10}$$

and $[A_S^*, H_{\Lambda,S}]$ consists of a finite (Λ-dependent) number of (otherwise uniformly bounded in S) terms, containing commutators of type

$$\left[\frac{S_i^\alpha}{S}, \frac{S_j^\beta}{S}\right] = \mathrm{i}\delta_{ij}\left(\frac{S_i^\gamma}{S}\right)\frac{1}{S},$$

which tend to zero in norm as $S \to \infty$. Since $A_S^*, B_S, H_{\Lambda,S}$ are uniformly (in S) bounded in norm, and

$$|\langle O\rangle_\Lambda^{\beta,S} \leqslant \|O\|$$

for an arbitrary operator O on $\mathscr{H}$, (2.6) follows from (2.8), (2.9) and (2.10).

Remark 2.1. We now establish the link between this section and §§ 3 and 4, especially § 3.

In § 3, the correspondence between the quantum and classical proofs of absence of phase transition with non-zero order parameter in one- and two-dimensional classical spin systems is shown in detail and—in part for the purpose of clarity—in a more general setting. The explicit connexion with the above result is as follows. We observe that the basic inequalities used in the classical proof may be obtained from the quantum inequalities by the limiting process indicated in the proposition. In fact, two inequalities are used there:

$$[(A, B)_\Lambda^\beta]^2 \leq (A, A)_\Lambda^\beta (B, B)_\Lambda^\beta \tag{2.11}$$

and

$$(A, A)_\Lambda^\beta \leq \tfrac{1}{2}\beta\langle AA^* + A^*A\rangle_\Lambda^\beta. \tag{2.12}$$

The point is that (2.12) reduces to an identity in the classical limit, while (2.11) reduces to the Schwartz inequality for classical functions on $\mathcal{H}_\Lambda^c$. But the Schwartz inequality for suitable operators (plus symmetry) is the only ingredient of the classical proof given in § 3.

The symmetry considerations are quite analogous for the classical and quantum cases. This point is shown in detail in § 3. Finally, in § 4 we remark that proposition 2.1 may be used to show that, similarly, the classical proof (Fröhlich *et al* 1976a, b) of existence of a phase transition follows directly from the quantum existence proof (Dyson *et al* 1976, 1977), in the case of the Heisenberg model with nearest-neighbour interactions.

3. Absence of phase transition for a class of two-dimensional systems

It has been shown that the Heisenberg model, in both classical and quantum forms, cannot exhibit a spontaneous magnetization at any finite temperature (Mermin 1967, Mermin and Wagner 1966). The proofs given rely mainly on two different tools, the intrinsic symmetry of the system and the Schwartz inequality for a well chosen non-degenerate sesquilinear form.

In Romerio and Vuillermot (1974) or Vuillermot and Romerio (1975) it has been shown that a generalized form of the classical Bogoliubov inequality can be derived for all systems whose configuration manifold is a compact connected real Lie group G and can be used to rule out the existence of a non-zero 'order parameter' at any finite temperature in a class of one- and two-dimensional systems defined by G-invariant Hamiltonians.

Our purpose, in this section, is to show that in the quantum case similar arguments can be developed and lead to an inequality which is the exact counterpart of the one used in Vuillermot and Romerio (1975) and Romerio and Vuillermot (1974).

In order to underline the analogy between the two cases, we briefly summarize the main steps of the proof for the classical case. Let, as previously, $\mathbb{Z}^\nu$ be a ν-dimensional lattice and Λ be a subset of $\mathbb{Z}^\nu$. We associate with each site $R \in \Lambda$ a copy G_R of a connected compact real Lie group G, of dimension n.

Let

$$G(\Lambda) \equiv \bigotimes_{R\in\Lambda} G_R \tag{3.1}$$

be the configuration manifold and $H_\Lambda \in C^\infty(G(\Lambda), \mathbb{R})$ be the Hamiltonian of the system.

Let

$$Z(\Lambda) = \int_{G(\Lambda)} \mathrm{d}g \exp(-\beta H_\Lambda(g)) \tag{3.2}$$

with $\beta=(kT)^{-1}$, be the partition function, and, for each $\varphi\in C^{\infty}(G(\Lambda),\mathbb{C})$

$$\langle\varphi\rangle_{\Lambda}=Z(\Lambda)^{-1}\int_{G(\Lambda)}\mathrm{d}g\,\varphi(g)\exp(-\beta H_{\Lambda}(g)) \tag{3.3}$$

be the 'thermal average'.

We then introduce the following positive sesquilinear form on the vector space $F(\Lambda)$ of the continuous mapping from $G(\Lambda)$ into an n-dimensional space M:

$$B(f,h)=\int_{G(\Lambda)}(f(g),h(g))_M\,\mathrm{e}^{-\beta H_{\Lambda}(g)}\,\mathrm{d}g \tag{3.4}$$

with

$$(f(g),h(g))_M=\sum_{\alpha=1}^{n}\bar{f}_{\alpha}(g)h_{\alpha}(g).$$

The Cauchy–Schwartz inequality yields

$$|B(f,h)|^2\leqslant B(f,f)B(h,h)\qquad f,h\in F(\Lambda). \tag{3.5}$$

To establish the Bogoliubov inequality, the special symmetry of the configuration manifold (3.1) is required; it is expressed in the following lemma.

Lemma 3.1. Let $(D_{\alpha})_{1\leqslant\alpha\leqslant n}$ be a family of differential operators on $G(\Lambda)$, and $(\phi_{\alpha})_{1\leqslant\alpha\leqslant n}$ a family of functions in $C^{\infty}(G(\Lambda),\mathbb{C})$. Then, for every family $(X_{\alpha})_{1\leqslant\alpha\leqslant n}$ of *left-invariant* complex vector fields on $G(\Lambda)$ we have

$$\beta\sum_{\alpha}\langle\bar{X}_{\alpha}(X_{\alpha}H_{\Lambda})\rangle\sum_{\beta}\langle|D_{\beta}\phi_{\beta}|^2\rangle\geqslant\left|\sum_{\alpha}\langle X_{\alpha}(D_{\alpha}\phi_{\alpha})\rangle\right|^2. \tag{3.6}$$

This lemma follows from (3.5) by choosing $f\equiv(f_{\alpha})_{\alpha=1}^{n}$ and $h\equiv(h_{\alpha})_{\alpha=1}^{n}$ appropriately (Vuillermot and Romerio 1975, Romerio and Vuillermot 1974).

It is important to note that (3.6) does not hold for an arbitrary vector field.

The next step is to choose X_{α}, D_{α} and ϕ_{α} for all α in such a way that the right-hand side of (3.6) be proportional to the 'order parameter'.

Let $\mathscr{A}_R$ be the Lie algebra of G_R. Because G_R is compact, there exists a strictly positive bilinear form B on $\mathscr{A}_R\times\mathscr{A}_R$, invariant under $\mathrm{Ad}(G_R)$, the adjoint representation of G_R; if $(X_{\alpha})_{1\leqslant\alpha\leqslant n}$ is a basis of $\mathscr{A}_R$ and $(Y_{\alpha}^{R})_{1\leqslant\alpha\leqslant n}$ is the dual basis with respect to B, the element

$$\gamma^{R}=\sum_{\alpha=1}^{n}X_{\alpha}^{R}Y_{\alpha}^{R} \tag{3.7}$$

called the Casimir element, belongs to the centre of $\mathrm{U}(\mathscr{A}_R)$, the universal enveloping algebra of $\mathscr{A}_R$. This result implies in particular that all spherical functions of G_R, defined with respect to a closed subgroup are eigensolutions of γ^{R}.

Making use of this standard result, we define

$$\begin{aligned}X_{\alpha}&=\sum_{R\in\Lambda}\exp(\mathrm{i}kR)X_{\alpha}^{R}\equiv X_{\alpha}(k)\\ D_{\alpha}&=\sum_{R\in\Lambda}\exp(-\mathrm{i}kR)Y_{\alpha}^{R}\equiv Y_{\alpha}(k)\end{aligned} \tag{3.8}$$

where k belongs to the first Brillouin zone of the lattice $\mathbb{Z}^{\nu}$.

It may be noted that the definitions of (3.8) implicitly imply that $\mathbb{Z}^\nu$ is embedded in an Euclidean space E^ν which allows, through the scalar product, the definition of the Brillouin zone.

Up to here, nothing has been assumed for H_Λ. We choose it in the form

$$H_\Lambda(g) = H_\Lambda^0(g) + \lambda H_\Lambda^1(g) \tag{3.9}$$

where $g \in G(\Lambda)$. We require that

$$H_\Lambda^0(g^{-1}g_1, \ldots, g^{-1}g_{|\Lambda|}) = H_\Lambda^0(g_1, \ldots, g_{|\Lambda|}) \tag{3.10}$$

and

$$H_\Lambda^1(g) = \sum_{R \in \Lambda} \varphi_R(g_R) \tag{3.11}$$

where φ_R is a spherical function on G_R with respect to a compact subgroup K of $G(\Lambda)$, i.e. it is an element of $C^\infty(G(\Lambda), \mathbb{C})$, which is an eigenfunction of the Casimir element defined in (3.7). Using this fact and (3.8), it is then an easy matter to show that taking $\phi_\alpha = H_\Lambda^1$, for all α, we get

$$\sum_\alpha \langle X_\alpha(D_\alpha H_\Lambda^1)\rangle = \text{constant} \times \langle \mathrm{H}_\Lambda^1\rangle \tag{3.12}$$

and that, by adding terms in k and $-k$,

$$\begin{aligned}\sum_\alpha \langle X_\alpha(\bar{X}_\alpha H_\Lambda)\rangle \\ &\leqslant -2 \sum_{R,R'} \{1-\cos[k\,.\,(R-R')]\} \sum_\alpha \langle X_\alpha^R \bar{X}_\alpha^{R'}(H_\Lambda^0)\rangle \\ &\quad -2\lambda \sum_R \sum_\alpha \langle (X_\alpha^R \bar{X}_\alpha^R) H_\Lambda^1\rangle.\end{aligned} \tag{3.13}$$

In this last inequality, we made use of the fact that in an infinitesimal form (3.10) reads

$$\sum_R X_\alpha^R H_\Lambda^0 = 0 \qquad \text{for all } \alpha$$

By some trivial majoration and integration on the Brillouin zone (Vuillermot and Romerio 1975, Romerio and Vuillermot 1974), one finally gets

$$1 \geqslant \beta^{-1}\mu |\langle H_\Lambda^1\rangle|^2 \int \frac{\mathrm{d}^\nu k}{Ak^2 + B|\lambda|} \tag{3.14}$$

where μ, A and B are positive constants (A is finite for a reasonable choice of H, for example (3.2) in Vuillermot and Romerio 1975).

To prove the absence of ordering in the corresponding quantum case, we first notice that having to define the quantum analogue of some generalized classical lattice spin systems, we can assume that to H_Λ correspond an operator $\hat{H}_\Lambda$ on a Hilbert space $\mathscr{H} = \otimes_{R \in \Lambda} M_R$, where the M_R are copies of the same Hilbert space M, satisfying some condition similar to (3.10).

On the other hand, we remark that the spherical functions with which we formed H_Λ^1 can be considered as 'canonically associated' to some irreducible representation of G (Helgason 1962). In the same way, we ask that to H_Λ^1 correspond an operator $\hat{H}_\Lambda^1$

formed from a single component of an irreducible tensor set. The elements of $\mathcal{H}$ are the states of the system so that we can replace (3.2) and (3.3) by

$$Z(\Lambda) = \mathrm{Tr}\, e^{-\beta \hat{H}_\Lambda} \tag{3.2'}$$

$$\langle \hat{A}_\varphi \rangle = Z(\Lambda)^{-1}\, \mathrm{Tr}(\hat{A}_\varphi\, e^{-\beta \hat{H}_\Lambda}) \tag{3.3'}$$

where $\hat{A}_\varphi$ is the linear operator corresponding to the observable φ.

To get the inequality corresponding to (3.4), let $\hat{F}(\Lambda)$ be the space of linear operators on $\oplus_1^n \mathcal{H}_\alpha$ where $\mathcal{H}_\alpha \equiv \mathcal{H}$.

With the help of the scalar product (2.1), we define on $\hat{F}(\Lambda)$ a positive sesquilinear form by

$$B(\hat{A}, \hat{B}) = \int_0^\beta \mathrm{d}\lambda \sum_\alpha \langle e^{\lambda \hat{H}_\Lambda} \hat{A}_\alpha\, e^{-\lambda \hat{H}_\Lambda} \hat{B}_\alpha \rangle \tag{3.4'}$$

where $\hat{A}_\alpha$ and $\hat{B}_\alpha$ are linear operators on $\mathcal{H}_\alpha$.

In analogy with § 2, we are inclined to think that (3.4) can be obtained from (3.4') by a limiting procedure corresponding to the passage from the quantum to the classical case. Using the Cauchy–Schwartz inequality for the above form and operators $A \equiv (\hat{A}_\alpha)_{\alpha=1}^n$ and $\hat{B} \equiv (\hat{B}_\alpha)_{\alpha=1}^n$ and following Ruelle (1969), we get

$$\left|\sum_\alpha \langle [\hat{C}_\alpha^*, \hat{A}_\alpha] \rangle\right|^2 \leq \tfrac{1}{2}\beta \sum_\alpha \langle \hat{A}_\alpha \hat{A}_\alpha^* + \hat{A}_\alpha^* A_\alpha \rangle \sum_\gamma \langle [\hat{C}_\gamma^*, [\hat{H}_\Lambda, \hat{C}_\gamma]] \rangle \tag{3.15}$$

where $\hat{C}_\alpha$ is defined in terms of $\hat{B}_\alpha$, $\alpha = 1, \ldots, n$, by

$$\hat{B}_\alpha = [\hat{C}_\alpha^*, \hat{H}_\Lambda].$$

It is evident that the expressions (3.4') and (3.15) are not valid for all operators on $\mathcal{H}$. We restrict ourselves here to bounded linear operators on $\mathcal{H}$.

Let U be a linear representation of G_R in M_R and $(\hat{T}_i^R)_{1 \leq i \leq n}$ an irreducible tensor set for G_R. We then have

$$U(g_R)\hat{T}_i^R U(g_R)^{-1} = \sum_{j=1}^m D_{i,j}^s(g_R)\hat{T}_j^R \tag{3.16}$$

where D^s is an m-dimensional irreducible representation of G_R, characterized by the discrete index s.

Writing

$$U(g_R) = \exp\left(\sum_1^n t_\alpha \hat{X}_\alpha^R\right) \qquad t_\alpha \in \mathbf{R}$$

and

$$D^s(g_R) = \exp\left(\sum_1^n t_\alpha a_\alpha^{R,s}\right)$$

where $(\hat{X}_\alpha)_{1 \leq \alpha \leq n}$ are the infinitesimal operators of U representing the elements $(X_\alpha^R)_{1 \leq \alpha \leq n}$ of the basis of $\mathcal{A}_R$ previously introduced, and the $(a_\alpha^{R,s})_{1 \leq \alpha \leq n}$ represent the same elements in the representation of $\mathcal{A}_R$ generated by D^s. We then have

$$[\hat{X}_\alpha, \hat{T}_i^{R,s}] = \sum_j (a_\alpha^{R,s})_{ij} \hat{T}_j^{R,s}. \tag{3.17}$$

If instead of the basis $(X_\alpha^R)_{1\leq\alpha\leq n}$ we take the basis $(Y_\alpha^R)_{1\leq\alpha\leq n}$, the elements corresponding to $\hat{X}_\alpha^R$ and $a_\alpha^{R,s}$ will be called $\hat{Y}_\alpha^R$ and $b_\alpha^{R,s}$ respectively.

Making use of (3.17), we then have

$$\sum_\alpha [\hat{Y}_\alpha^R[\hat{X}_\alpha^R, \hat{T}_i^{R,s}]] = \sum_{\alpha=1}^n \sum_{j,k=1}^m (a_\alpha^{R,s})_{i,k}(b_\alpha^{R,s})_{k,j}\hat{T}_j^{R,s}.$$

But, as is well known in representation theory (Bourbaki 1960), the element

$$\hat{\gamma}_{i,j}^{R,s} = \sum_{\alpha=1}^n \left(\sum_{k=1}^m (a_\alpha^{R,s})_{i,k}(b_\alpha^{R,s})_{k,j}\right) \tag{3.7$'$}$$

is the Casimir element of the representation D^s and is proportional to the identity in the representation space. Let us now introduce the operator

$$\hat{A}_\alpha^{R,s} = \mathbb{1}\otimes\ldots\otimes\mathbb{1}\otimes[\hat{X}_\alpha^R, \hat{T}_0^{R,s}]\otimes\mathbb{1}\otimes\ldots\otimes\mathbb{1}$$

where $\hat{T}_0^{R,s}$ is a particular element of the tensor set; $\hat{T}_0^{R,s}$ is supposed to be Hermitian. We then define

$$\hat{A}_\alpha^s = \sum_{R\in\Lambda} \exp(\mathrm{i}k\,.\,R)\hat{A}_\alpha^{R,s} \tag{3.8$'$}$$

and

$$\hat{C}_\alpha^* = \sum_{R\in\Lambda} \exp(-\mathrm{i}k\,.\,R)\hat{Y}_\alpha^{R,s}$$

where $\hat{Y}_\alpha^R$ is identified with $\mathbb{1}\otimes\ldots\otimes\hat{Y}_\alpha^R\otimes\ldots\otimes\mathbb{1}$.

As before k belongs to the first Brillouin zone of $\mathbb{Z}^\nu$ and $k\,.\,R$ is the scalar of the two elements.

For the Hamiltonian, we impose a form similar to (3.9). In an infinitesimal form (3.10) becomes

$$\left[\sum_{R\in\Lambda} \hat{X}_\alpha^R, \hat{H}_\Lambda^0\right] = 0. \tag{3.10$'$}$$

We also put

$$\hat{H}_\Lambda^1 = \sum_{R\in\Lambda} \hat{T}_0^{R,s} \tag{3.11$'$}$$

where $\hat{T}_0^{R,s}$ is identified with $\mathbb{1}\otimes\ldots\otimes\hat{T}_0^s\otimes\ldots\otimes\mathbb{1}$, that is to an element which depends on R only by its position on the lattice.

We can now make the analogy between (3.15) and (3.6) completely transparent. We write

$$\hat{\mathscr{X}}_\alpha\hat{A}_\alpha \equiv [\hat{C}_\alpha, \hat{A}_\alpha]$$

and define

$$\hat{\phi}_\alpha = H_\Lambda^1$$

for all α. Then $\hat{A}_\alpha^s$ defined by (3.8$'$) may be written

$$\hat{A}_\alpha^s = \mathrm{i}\hat{D}_\alpha^0\hat{\phi}_\alpha$$

where

$$\mathrm{i}\hat{D}^0_\alpha(\hat{\phi}_\alpha) \equiv \left[\sum_{R\in\Lambda} \exp(\mathrm{i}k \,.\, R)\hat{X}^R_\alpha, \hat{\phi}_\alpha\right].$$

$\hat{\mathscr{X}}_\alpha$ and $\hat{D}^0_\alpha$ are clearly the precise analogues of the derivations (3.8). In this notation, (3.15) becomes

$$\left|\sum_\alpha \langle \hat{\mathscr{X}}_\alpha(\hat{D}^0_\alpha\hat{\phi}_\alpha)\rangle\right|^2 \leqslant \tfrac{1}{2}\beta \sum_\alpha \langle(\hat{D}^0_\alpha\hat{\phi}_\alpha)(D^0_\alpha\hat{\phi}_\alpha)^* + (\hat{D}^0_\alpha\hat{\phi}_\alpha)^*(D^0_\alpha\hat{\phi}_\alpha)\rangle \sum_\gamma \langle \hat{\mathscr{X}}^*_\gamma(\hat{\mathscr{X}}_\gamma\hat{H}_\Lambda)\rangle. \tag{3.6$'$}$$

To exploit the analogy further, we use (3.7$'$), (3.8$'$) and (3.17) to get

$$\begin{aligned}\sum_\alpha [C^*_\alpha, A^s_\alpha] &= \sum_{R,R'\in\Lambda} \exp[\mathrm{i}k \,.\, (R-R')] \sum_{\alpha=1}^{n} [\hat{Y}^R_\alpha, [\hat{X}^R_\alpha, \hat{T}^{R,s}_0]]\\ &= \sum_R \sum_\alpha \sum_{j,k} (b^{R,s}_\alpha)_{j,k}(a^{R,s}_\alpha)_{0,j}\hat{T}^R_K = \mu^s\hat{T}^s_0\end{aligned}$$

and finally

$$\left|\sum_\alpha \langle[C^*_\alpha, A^s_\alpha]\rangle\right|^2 = |\mu^s|^2|\langle\hat{T}^s_0\rangle|^2 \tag{3.12$'$}$$

where μ^s is a constant.

Adding terms in k and $-k$ and making use of (3.10$'$), we also have

$$\begin{aligned}\sum_j \langle[\hat{C}^*_j, [\hat{H}_\Lambda, \hat{C}_j]]\rangle &\leqslant -2\sum_{R,R'} \{1-\cos[k \,.\, (R-R')]\} \sum_\alpha \langle\langle[\hat{Y}^R_\alpha, [\hat{H}^0_\Lambda, Y^{R'*}_\alpha]]\rangle\rangle\\ &\quad -2\lambda \sum_R \sum_\alpha \langle[\hat{Y}^R_\alpha, [\hat{H}^1_\Lambda, \hat{Y}^{R*}_\alpha]]\rangle\end{aligned}$$

but

$$\langle[\hat{Y}^R_\alpha, [\hat{H}^1_\Lambda, \hat{Y}^{R*}_\alpha]]\rangle = \sum_{i,j} (b^{R,s}_\alpha)_{ji}(\overline{b^{R,s}_\alpha})_{0j}\langle\hat{T}^{R,s*}_i\rangle$$

as

$$[T^R_0, Y^{R*}_\alpha] = [Y^R_\alpha, T^R_0]^* = \sum_j (\overline{b^{R,s}_\alpha})_{0j}(\hat{T}^{R,s}_j)^*.$$

Noticing that $1-\cos x \leqslant \frac{1}{2}x^2$ and summing both sides of the inequality (3.15) on the first Brillouin zone, we get, after standard majoration and after taking the thermodynamic limit, $\Lambda\to\infty$, a formula similar to (3.14), in which A is finite for a class of H^0_Λ defined by the condition

$$\lim_{\Lambda\to\infty} \sum_{R,R'\in\Lambda} |\langle[\hat{X}^R_\alpha, [\hat{H}^0_\Lambda, \hat{Y}^{R'*}_\alpha]]\rangle|(R-R')^2 < \infty.$$

The result is thus essentially the same as for the classical case.

4. Remarks on the relation between classical and quantum existence proofs

In Dyson *et al* (1976, 1977) a proof of the existence of a phase transition for the quantum mechanical isotropic Heisenberg model with nearest-neighbour interactions, spin S arbitrary, and in any number $\nu \geq 3$ dimensions, was sketched. The Hamiltonian for the region $\Lambda \subset \mathbb{Z}^\nu$ was given by

$$H_\Lambda = \sum_{\alpha \in \Lambda} \sum_{i=1}^{\nu} (S^2 - \boldsymbol{S}_\alpha \cdot \boldsymbol{S}_{\alpha+\delta_i}) \tag{4.1}$$

on $\mathcal{H}_\Lambda = \otimes_{i=1}^{|\Lambda|} \mathbb{C}_i^{2S+1}$, $S_\alpha^{(j)}$, $j = 1, 2, 3$, being the spin operators for the ith lattice site, and $|\Lambda|$ is the number of sites in Λ. The proof was based upon the inequality

$$\sum_{j=1}^{3} (S_p^{(j)}, S_{-p}^{(j)})_\Lambda^\beta \leq \frac{3}{2\beta E_p} \qquad p \neq 0, p \in B_1 \tag{4.2}$$

where $E_p \equiv \nu - \Sigma_{j=1}^{\nu} \cos p_j$, $(A, B)_\Lambda^\beta$ was defined in § 2, and

$$S_p^{(i)} \equiv |\Lambda|^{-\frac{1}{2}} \sum_{\alpha \in \Lambda} S_\alpha^{(j)} \, \mathrm{e}^{\mathrm{i}p \cdot \alpha} \tag{4.3}$$

$$B_1 \equiv [-\pi, \pi]^\nu.$$

Proposition 2.1. As a consequence of (4.2), the classical Heisenberg model corresponding to (4.1) undergoes a phase transition.

Proof. From (4.2), if $\beta = \beta'/S^2$

$$\frac{1}{S^2} \sum_{j=1}^{3} (S_p^{(j)}, S_{-p}^{(j)})_\Lambda^{\beta'/S^2} \leq \frac{3}{2\beta' E_p} \qquad p \neq 0, p \in B_1. \tag{4.4}$$

Now taking the limit ($S \to \infty$) for fixed Λ, and using (2.6), we find

$$\sum_{j=1}^{3} \langle t_p^{(j)}, t_{-p}^{(j)} \rangle_\Lambda^{\beta'} \leq \frac{3}{2\beta' E_p} \qquad p \neq 0, p \in B_1 \tag{4.5}$$

where $\boldsymbol{t}_p$ are the Fourier transforms of the classical unit vectors defined as in (4.3). (4.5) is the relation established in Fröhlich *et al* (1976a, b), which, together with Parseval's equality, proves the existence of a phase transition (see Fröhlich *et al* 1976a, b).

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